# 1  Filters

It quickly becomes apparent that sequences are not adequet in a general topological setting and that new more general concepts need to be introduced. Two such generalisation are common: nets and filters. Filters are one of the fundamental objects in a Convergence Space. However, their use it not limited to Convergence Spaces and they are interesting in their own right in a Topological setting. In this section we introduce the concept of a filter and illustrate some applications to Point-Set Topology working towards elegant proof of Tychonoff's Theorem. Throughout this section all our spaces will be Topological, any further structure will be clearly indicated. With that said, the definitions in "Basic Concepts" extend through to more general spaces.

## 1.1  Basic Concepts

**Definition 1.1.** A nonvoid family $\mathcal{F}$ of subsets $F$ of a set $X$ is called a **filter** in $X$ if it satisfies the following axioms:

(F. 1)  *If $F \in \mathcal{F}$ then $F$ is not void*

(F. 2)  *If $F_1 \in \mathcal{F}$ and $F_1 \subseteq F_2$ then $F_2 \in F$*

(F. 3)  *If $F_1, F_2 \in \mathcal{F}$ and $F_1 \cap F_2 \in \mathcal{F}$*

It is useful to sumarise the properties of a filter as follows: a filter is a nonvoid family of nonvoid subsets of X which have the finite intersection property and are closed under supersets.

**Remark 1.2.** The Axiom (F. 3) can be weakened as follows:

(F. 3a)  *If $F_1, F_2 \in \mathcal{F}$ then there exists an $F \in \mathcal{F}$ such that $F \subseteq F_1 \cap F_2$*

Clearly (F. 2) and (F. 3a) imply (F. 3). However there is no advantage for us to take (F. 3a) as one of our Axioms, so we retain (F. 1-3) as our Axioms.

**Remark 1.3.** When visualising Filters it is often useful to consider an equivalent form of (F. 2):

(F. 2a)  *If $F_1 \in \mathcal{F}$ and $F_2$ is a subset of $X$, then $F_1 \cup F_2 \in F$*

While not practical, it does shed more light on the nature of filters. One consequence of (F. 2a) is that if we are given a sufficient number of elements from the filter, we are able to generate the rest of the filter by taking supersets. Together with (F. 3a), this useful characterisation of a filter is captured in the following definition:

**Definition 1.4.** A nonvoid family $\mathcal{B}$ of subsets $B$ of $X$ is called a **filter base** in $X$ if it satisfies the following axioms:

(B. 1)  *If $B \in \mathcal{B}$ then $B$ is nonvoid*

(B. 2) *If $B_1$, $B_2 \in \mathcal{B}$ then there exists a $B \in \mathcal{B}$ such that $B \subseteq B_1 \cap B_2$*

The filter $\mathcal{F}$ generated by the filter base $\mathcal{B}$ is the family of those sets $F \subseteq X$ which contain some $B \in \mathcal{B}$

**Remark 1.5.** Every filter $\mathcal{F}$ is also a filter base and the filter is generates is $\mathcal{F}$

Usually we define filters by defining what sets are in their filter base. As we will soon see, it is often far easier to work with filter bases than filters themselves and they also give greater insight to the "inner-workings" of the filter in question. It is also possible to define a filter subbase but for our purposes they will not be needed.

**Definition 1.6.** We say a filter is **countable** if it admits a countable base

To complete the generalisation of sequences to filters we make the following definition:

**Definition 1.7.** Let $(x_n)_{n=1}^{\infty}$ be a sequence in the set X and let $\mathcal{F}$ be the filter generated by the denumerable base $\mathcal{B} = \{B_n\}$ where $B_n = \{x_v : v \geq n\}$. Then $\mathcal{F}$ is called the **elementary filter** generated by the sequence $(x_n)$.

**Remark 1.8.** The elementary filter for a sequence $(x_n)$ is the set of all subsets $F$ of $X$ which contain all but a finitely many elemets of the sequence $(x_n)$

We will later show that the elementary filter captures the same information as the sequence used to generate it.
We can create a partial ordering on the set of all filters in the following way:

**Definition 1.9.** We say a filter $\mathcal{F}$ is **finer** than a filter $\mathcal{G}$ if for every $G \in \mathcal{G}$ there is an $F \in \mathcal{F}$ such that $F \subseteq G$. In symbols we say $\mathcal{F} \geq \mathcal{G}$ or $\mathcal{G} \leq \mathcal{F}$. We also sometimes say that $\mathcal{G}$ is **coarser** than $\mathcal{F}$ if $\mathcal{F}$ is finer than $\mathcal{G}$.

**Definition 1.10.** We say a filter $\mathcal{F}$ is **strictly finer** than a filter $\mathcal{G}$ if for every $G \in \mathcal{G}$ there is an $F \in \mathcal{F}$ such that $F \subset G$. In symbols we say $\mathcal{F} > \mathcal{G}$ or $\mathcal{G} < \mathcal{F}$. We also sometimes say that $\mathcal{G}$ is **strictly coarser** than $\mathcal{F}$ if $\mathcal{F}$ is strictly finer than $\mathcal{G}$.

**Lemma 1.11.** *The relation $\geq$ defines a partial order on the set of all filters.*

*Proof.* Reflexivity is clear. To show antisymmetry suppose $\mathcal{F} \geq \mathcal{G}$ and $\mathcal{G} \geq \mathcal{F}$ and let $F \in \mathcal{F}$. Then $\exists\, G \in \mathcal{G}$ such that $G \subseteq F$ so by (F. 2) $F \in \mathcal{G}$. Similarly, if $G \in \mathcal{G}$ then $\exists\, F \in \mathcal{F}$ such that $F \subseteq G$ so $G \in \mathcal{F}$. Thus $\mathcal{F} = \mathcal{G}$ and $\geq$ is antisymmetric. To show transivity suppose $\mathcal{F} \geq \mathcal{G}$ and $\mathcal{G} \geq \mathcal{H}$. Then for every $H \in \mathcal{H}$ $\exists\, G \in \mathcal{G}$ such that $G \subseteq H$ and $\exists\, F \in \mathcal{F}$ such that $F \subseteq G$ so $F \subseteq G \subseteq H$ so $F \subseteq H$ and $F \geq \mathcal{H}$. Hence, $\geq$ is transitive and thus a partial ordering on the set of all filters. □

**Definition 1.12.** An **ultrafilter** $\mathfrak{U}$ in a set X is a filter such that there exists no filter in $X$ which is strictly finer than $\mathfrak{U}$.

**Remark 1.13.** This definition is equivalent to saying that $\mathfrak{U}$ is maximal with respect to the partial ordering $\geq$

**Theorem 1.14.** *Given any filter $\mathcal{F}$ in a set $X$ there is an ultrafilter $\mathfrak{U}$ in $X$ which is finer than $\mathcal{F}$*

*Proof.* We invoke Zorn's Lemma: Let $\mathfrak{M}$ be the set of all filters $\mathcal{G}$ finer than $\mathcal{F}$. Now $\mathfrak{M} \neq \varnothing$ since $\mathcal{F} \in \mathfrak{M}$. Then $\geq$ is a partial order on $\mathfrak{M}$. Every linearly ordered subfamily $\mathfrak{L}$ of $\mathfrak{M}$ has an upper bound in $\mathfrak{M}$, namely, $\cup\{\mathcal{G} : \mathcal{G} \in \mathfrak{L}\}$ is a filter in $X$ which is finer than every $\mathcal{G} \in \mathfrak{L}$. Hence Zorn's Lemma applies and so there is a maximal element $\mathfrak{U} \in \mathfrak{M}$. No other filter $\mathcal{G}$ is finer than $\mathfrak{U}$ since if $\mathfrak{U} \leq \mathcal{G}$ then $\mathcal{F} \leq \mathcal{G}$ and so $\mathcal{G} \in \mathfrak{M}$ and hence $\mathcal{G} = \mathfrak{U}$. $\qquad\square$

**Theorem 1.15.** *If $A_1 \cup \cdots \cup A_n \in \mathfrak{U}$, then $A_i \in \mathfrak{U}$ for some $i$.*

*Proof.* It suffices to treat the case $n = 2$. Suppose $A_1 \notin \mathfrak{U}$ but $A_1 \cup A_2 \in \mathfrak{U}$ and consider the family $\mathcal{G}$ of those sets $G$ which have the property that $A_1 \cup G \in \mathfrak{U}$. Since $A_2 \in \mathcal{G}$, $\mathcal{G} \neq \varnothing$ and none of it's elements are void because $A_1 \notin \mathfrak{U}$. This proves that $\mathcal{G}$ saistfies (F. 1). (F. 2-3) are easily to verify so $\mathcal{G}$ is a filter in $X$. By (F. 2) we have that $\mathfrak{U} \leq \mathcal{G}$ but since $\mathfrak{U}$ is an ultrafilter $\mathcal{G} \leq \mathfrak{U}$ and so $A_2 \in \mathcal{G} \Rightarrow A_2 \in \mathfrak{U}$. $\qquad\square$

**Corollary 1.16.** *Let $\mathcal{F}$ be a filter on $X$. Then $A \in \mathcal{F}$ or $X \backslash A \in \mathcal{F}$, $\forall A \subseteq X$ if and only if $\mathcal{F}$ is an ultrafilter.*

*Proof.* ($\Rightarrow$) Let $\mathcal{F}$ be a filter with the given property and let $\mathcal{F} \leq \mathcal{G}$ for some fitler $\mathcal{G}$ in $X$. If $G \in \mathcal{G}$, then $G \in \mathcal{F}$ or $X \backslash G \in \mathcal{F}$. But $X \backslash G \in \mathcal{F}$ would imply that $G \cap X \backslash G = \varnothing \in \mathcal{G}$. Thus, $G \in \mathcal{F}$ and so $\mathcal{G} \leq \mathcal{F}$ and hence $\mathcal{F} = \mathcal{G}$ and $\mathcal{F}$ is maximal i.e. an ultrafilter.
($\Leftarrow$) Follows from Theorem 1.15 $\qquad\square$

Ultrafilters are very useful tools in point-set topology so long as we are willing to accept the Axiom of Choice. We will investigate ultrafilters in greater detail in later sections as any further investigation would detract from the expository nature of this section. We conclude this section with two important types of filters.

**Definition 1.17.** If $A \subset X$ then the family $(A)_\bullet = \{B \subseteq X : A \subseteq B\}$ is a filter called the **principle filter** of A. The set $\{A\}$ is a base of the principle filter of A.

Simply put, the principle filter of $A$ is the family of subsets of $X$ which contain $A$. We will make great use of the principle ultrafilter at a point $x$ denoted $(x)_\bullet$ or more commonly, $\dot{x}$ which is the set of all subsets of $X$ which contain $x$.

**Definition 1.18.** If a filter $\mathcal{F}$ on a set $X$ satisfies $\bigcap_{F \in \mathcal{F}} F = \varnothing$ then we say that $\mathcal{F}$ is a **free filter**.

## 1.2 Convergence in Topological Spaces

We begin exploring the applications of filters to point-set topology. Some of the concepts presented here are not general enough for a convergence space and hence some of these concepts will be redefined later for the more general setting. Obviously, the generalisations and the definitions presented here will agree in a topological setting. This section will introduce the adherence and limits points of a filter and present some applications.

**Definition 1.19.** The **neighbourhood filter** at a point $x$, denoted $\mathcal{N}(x)$ is the filter generated by family of open sets containing $x$.

**Definition 1.20.** The **adherence** of a filter $\mathcal{F}$ in a topological space $X$ is the closed set

$$\text{adh } \mathcal{F} = \cap \{\overline{F} : F \in \mathcal{F}\}$$

Where $\overline{F}$ denotes the topological closure of F. The elements of adh $\mathcal{F}$ are called the **adherence points** of the filter $\mathcal{F}$.

**Remark 1.21.** The adherence of $\mathcal{F}$ can be a void set. Define a family of subsets of $\mathbb{N}$ as follows: $\mathcal{F} = \{F \subseteq \mathbb{N} : \text{card}(\mathbb{N} \backslash F) < \infty\}$

**Claim 1.21.1.** $\mathcal{F}$ is a filter

*Proof.* Since $\mathbb{N} \in \mathcal{F}$, $\mathcal{F}$ is nonvoid, so we verify the axioms (F. 1-3)

(F. 1) If $\varnothing \in \mathcal{F}$ then $\text{card}(\mathbb{N} \backslash \varnothing) < \infty$. But $\mathbb{N} \backslash \varnothing = \mathbb{N}$ which has infinite cardinality. Thus $\varnothing \notin \mathcal{F}$

(F. 2) Let $F_1 \in \mathcal{F}$ and $F_1 \subseteq F_2$. Then $\text{card}(\mathbb{N} \backslash F_2) \leq \text{card}(\mathbb{N} \backslash F_1) < \infty \Rightarrow F_2 \in \mathcal{F}$. Hence $\mathcal{F}$ is closed under supersets.

(F. 3) Let $F_1, F_2 \in \mathcal{F}$. $\mathbb{N} \backslash (F_1 \cap F_2) = \mathbb{N} \backslash F_1 \cup \mathbb{N} \backslash F_2$ and $\text{card}(\mathbb{N} \backslash F_1 \cup \mathbb{N} \backslash F_2) \leq \text{card}(\mathbb{N} \backslash F_1) + \text{card}(\mathbb{N} \backslash F_2) < \infty$. So $F_1 \cap F_2 \in \mathcal{F}$ and hence $\mathcal{F}$ is closed under finite intersections

$\square$

**Claim 1.21.2.** $\mathcal{F}$ has void adherence

*Proof.* Suppose $x \in \text{adh } \mathcal{F}$ then $x \in \overline{F} \; \forall F \in \mathcal{F}$ and in particular $x \in \mathbb{N}$. The set $\overline{\mathbb{N} \backslash [1, x + 1]}$ does not contain $x$ and is an element of $\mathcal{F}$ and thus $x$ cannot be an adherence point of $\mathcal{F}$. Thus $\mathcal{F}$ has void adherence. $\square$

It is worth noting that by our definition, every filter with void adherence is a free filter but not very free filter has void adherence. This contrasts with some authors who define a free filter as a filter with void adherence. However, the difference will be of importance when we are working with convergence spaces.

**Lemma 1.22.** *$x$ is an adherence point of $\mathcal{F}$ if and only if $F \cap N_x \neq \varnothing$ for every set $F \in \mathcal{F}$ and for every neighbourhood $N_x$ of $x$.*

*Proof.* This follows from the fact that $x \in \overline{F} \Leftrightarrow F \cap N_x \neq \varnothing$ for every neighbourhood $N_x$ of $x$. $\qquad\square$

**Lemma 1.23.** *If $\mathcal{B}$ is a base for the filter $\mathcal{F}$ then adh $\mathcal{F} = \bigcap\{\overline{B} : B \in \mathcal{B}\}$*

*Proof.* Let $\mathcal{B}$ be a base for $\mathcal{F}$. Then $B \in \mathcal{B} \Rightarrow B \in \mathcal{F}$ so $\bigcap \overline{F} \subseteq \bigcap \overline{B}$ and since every $F$ contains some $B$ we also see that $\bigcap \overline{B} \subseteq \bigcap \overline{F}$ hence adh $\mathcal{F} = \bigcap\{\overline{B} : B \in \mathcal{B}\}$ $\qquad\square$

**Remark 1.24.** In light of the previous Lemma, it makes sense to talk about the adherence of a filter base defined in the obvious way. It is often easier to work with filter bases rather than filters when finding calculating adherences. As a demonstration we calculate the adherence of the elementary filter

**Example 1.25.** Let $(x_n)$ be a sequence in $X$ and let $\mathcal{F}$ be the elementary filter for $(x_n)$. By definition, $x \in$ adh $F \Leftrightarrow$ every neighbourhood $N_x$ intersects every $B_n = \{x_v : v \geq n\}$, that is if and only if every $N_x$ contains $x_v$ with $v \geq n$ where $n$ is arbitrary. Hence adh $\mathcal{F}$ consists of those points $x \in X$ whose neighbourhoods $N_x$ contain an infinity of terms of $(x_n)$. This is precisely the definition of an accumlation point of a sequence, hence the adherence of an elemntary filter is the set of accumlation points of the corresponding sequence.

**Definition 1.26.** Let $\mathcal{F}$ be a filter in a topological space X. A point $x \in X$ is called a **limit point** of $\mathcal{F}$ if every neighbourhood $N_x$ of $x$ contains some $F \in \mathcal{F}$. The set of all limit points is called the limit of the filter $\mathcal{F}$ and is denoted $\lim \mathcal{F}$

**Remark 1.27.** This definition is equivalent to saying $x \in \lim \mathcal{F} \Leftrightarrow \mathcal{F} \geq \mathcal{N}(x)$.

**Remark 1.28.** We will often use the notation $\mathcal{F} \longrightarrow x$ read "the filter $\mathcal{F}$ converges to $x$" instead of $x \in \lim \mathcal{F}$ as this notation is far more convienient when working with convergence structures.

**Remark 1.29.** It is clear that $\lim \mathcal{F} \subseteq$ adh $\mathcal{F}$

We can extend the definition of a limit point in the natural way to filter bases as follows:

**Definition 1.30.** Let $\mathcal{B}$ be a filter base in a topological space X. A point $x \in X$ is called a **limit point** of $\mathcal{B}$ if for every $N \in \mathcal{N}(x)$ there is a $B \in \mathcal{B}$ such that $B \subseteq N$.

**Lemma 1.31.** *The limit of a filter base $\mathcal{B}$ is identical with the limit of the filter generated by $\mathcal{B}$*

*Proof.* Since every $B \in \mathcal{B}$ is also an element of $\mathcal{F}$, it follows that $\lim \mathcal{B} \subseteq \lim \mathcal{F}$. Conversely, since every $F \in \mathcal{F}$ contains some $B \in \mathcal{B}$ it follows that $\lim \mathcal{F} \subseteq \lim \mathcal{B}$ and thus the limits are equal. $\qquad\square$

**Remark 1.32.** As before, calculating limits using filter bases is often much easier than using filters. We illustrate this by calculating the limit of the elementary filter. In doing so we see that the elementary filter captures the same information as the sequence does and hence is an appropriate generalisation.

**Example 1.33.** Let $(x_n)$ be a sequence in $X$ and $\mathcal{F}$ the corresponding elementary filter and $\mathcal{B}$ a filter base for the filter $\mathcal{F}$. By definition, $x \in \lim \mathcal{B} \Leftrightarrow$ for every $N \in \mathcal{N}(x)$ there is a $B \in \mathcal{B}$ such that $B \subseteq N$. So each $N$ contains some $B_n = \{x_v : v \geq n\}$ so every neighbourhood contains some n-tail of the sequence $(x_n)$ which is precisely the definition of a limit point of a sequence.

One of the key failings of sequences in general topological spaces is that the following lemma on sequences of real numbers cannot be extended to arbitrary topological spaces:

**Lemma 1.34.** *A point $x$ is an adherence point of a sequence $(x_n)$ of real numbers if and only if there is a subsequence of $(x_n)$ which is convergent to $x$.*

However, we can extend the key features of this Lemma to filters on arbitrary topological spaces:

**Lemma 1.35.** *A point $x$ is an adherence point of a filter $\mathcal{F}$ in $X$ if and only if there is a filter $X$ which is finer than $\mathcal{F}$ and is converging to $x$*

*Proof.* Let $x$ be an adherence point of a filter $\mathcal{F}$. Then $N_x \cap \mathcal{F} \neq \varnothing$ for every neighbourhood $N_x$ and for every $F \in \mathcal{F}$. Hence the family $\{N_x \cap F : N_x \in \mathcal{N}(x)$ and $F \in \mathcal{F}\}$ is a filter in $X$. This filter is finer than $\mathcal{F}$ and also of $\mathcal{N}(x)$ and it is convergent to $x$. This proves necessity. Next suppose that there is a filter $\mathcal{G}$ finer than $\mathcal{F}$ such that $x \in \lim \mathcal{G}$. Then $\mathcal{F} \leq \mathcal{G}$ and $\mathcal{N}(x) \leq \mathcal{G}$ so $F \cap N_x \in \mathcal{G}$ for any $F \in \mathcal{F}$ and $N_x \in \mathcal{N}$. Thus $F \cap N_x \neq \varnothing$ for any $F \in \mathcal{F}$ and $N_x \in \mathcal{N}(x)$ and so $x \in \overline{F}$. Since F is arbitrary we obtain that $x \in \text{adh } \mathcal{F}$. This concludes the proof. $\qquad \square$

**Remark 1.36.** In this light, we can see how the notion of a subsequence is generalised to the notion of a filter $\mathcal{G}$ being finer than a filter $\mathcal{F}$.

**Theorem 1.37.** *If $\mathfrak{U}$ is an ultrafilter in a topological space $X$, then $\text{adh } \mathfrak{U} = \lim \mathfrak{U}$*

*Proof.* If $x \in \text{adh } \mathfrak{U}$ then there is a filter finer than $\mathfrak{U}$ convergent to $x$, but $x$ is an ultrafilter so such a filter must be $\mathfrak{U}$ itself and the result follows. $\qquad \square$

Recall that a space is Hausdorff or $T_2$ if for all $x, y \in X$ with $x \neq y$ there exists open neighbourhoods $O_x$ of $x$ and $O_y$ of y such that $O_x \cap O_y = \varnothing$. We now demonstrate that this definition is equivalent to filters having at most one limit point in X.

**Theorem 1.38.** *A topological space $X$ is Hausdorff if and only if every filter in $X$ has at most one limit point*

*Proof.* Suppose $X$ is Hausdorff. Then let $x$ and $y$ be distinct points in $X$ and let $O_x$ and $O_y$ be the disjoint open neighbourhoods guarenteed by by the Hausdorff property of $x$ and $y$ respectively. If $x \in \lim \mathcal{F}$ then there is an $F \in \mathcal{F}$ such that $F \subseteq O_x$. Therefore $F \subseteq X \backslash O_y$ and $\overline{F} \subseteq X \backslash O_y$. This shows that $y \notin \bigcap \overline{F} = \text{adh } \mathcal{F}$. By $\lim \mathcal{F} \subseteq \text{adh } F$ we have that $\lim \mathcal{F} = \text{adh } \mathcal{F} = \{x\}$. This

shows that $\mathcal{F}$ has a unique limit.

Suppose now that filters have at most one limit point and suppose that $X$ is not Hausdorf. Then there are distinct points $x$ and $y$ in $X$ such that $N_x \cap N_y \neq \varnothing$ for every $N_x \in N(x)$ and $N_y \in N(y)$. Then $\mathcal{F} = \{N_x \cap N_y\}$ is a filter in $X$. Clearly $\mathcal{F} \geq \mathcal{N}(x)$ and $\mathcal{F} \geq \mathcal{N}(y)$ so $\lim \mathcal{F} = \{x, y\}$, a contradiction. $\qquad \square$

We can also use filters to characterise $T_3$ spaces as well. Recall that a $T_3$ space is a space with the property that given any closed set $A$ and any point $b \notin A$ there exists disjoint open sets $O_A$ and $O_b$ such that $A \subseteq O_A$ and $b \in O_b$. Firstly we need to introduce the closure of a filter:

**Definition 1.39.** The filter generated by the base $\{\overline{F} : F \in \mathcal{F}\}$ is called the **closure** of the filter $\mathcal{F}$ and is denoted $\overline{\mathcal{F}}$

**Corollary 1.40.** $\lim \overline{\mathcal{F}} \subset \lim \mathcal{F}$

*Proof.* This follows since $\mathcal{F} \geq \overline{\mathcal{F}}$ $\qquad \square$

**Theorem 1.41.** *Let $X$ be a topological space, then the following statements are equivalent*

1. *$X$ is a $T_3$ space.*

2. *$\lim \overline{\mathcal{F}} = \lim \mathcal{F}$ for every filter $\mathcal{F}$ in $X$*

*Proof.* $(1) \Rightarrow (2)$ Let $X$ be a $T_3$ space and let $x \in X$. Then given $O_x$ there is an open set $Q_x$ having the property that $x \in Q_x \subseteq \overline{Q}_x \subseteq O_x$. If $x \in \lim \mathcal{F}$ then there is an $F \in \mathcal{F}$ such that $F \subseteq Q_x$ and so $\overline{F} \subseteq \overline{Q}_x \subseteq O_x$. Therefore every open neighbourhood $O_x$ contains a set $\overline{F} \in \overline{\mathcal{F}}$ and so $x \in \lim \overline{\mathcal{F}}$.

$(2) \Rightarrow (1)$ Now suppose that $X$ is not a $T_3$ space but $\lim \overline{\mathcal{F}} = \lim \mathcal{F}$ for every filter $\mathcal{F}$ in $X$. Then there is a point $x$ and an open set $O_x$ containing $x$ with the following property: If $Q_x$ is an open set containing $x$, then $\overline{Q}_x$ is not contained in $O_x$. The family of subsets $\{Q_x Q_x \in \mathcal{N}(x)\}$ is a base for $\mathcal{N}(x)$ and no element of this base is contained in $O_x$. Thus $x \notin \lim \overline{\mathcal{N}(x)}$ but clearly $x \in \lim \mathcal{N}(x)$, a contradiction. $\qquad \square$

## 1.3 Application of Filters to the Riemann Integral

It is possible to rework the defintion and construction of the Riemann Integral using filters. This avoids some of the awkward definitions that are traditionally set up in order to define the integral. This section will introduce the Riemann Integral under this new framework, prove equivalence with the traditional framework and some use this characterisation to prove some basic theorems. As usual, we begin with a definition:

**Definition 1.42.** A **partition** of an interval $I = [a, b]$ is a finite sequence of points $(x_n)$ such that

$$a = x_0 < x_1 < \cdots < x_n = b$$

A **refinement** of a partition $P$ is another partition $Q$ of the same interval which contains all the points of $P$ and possibly some additional points. In this case $Q$ is said to be finer than $P$.

**Definition 1.43.** We define the **upper sum** of a function $f$ on $I$ with respect to a partition $P$ as

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i$$

Where $M_i = \sup f(x)$ such that $x_{i-1} \le x \le x_i$ and $\Delta x_i = |x_i - x_{i-1}|$. We define the lower sum $L(f, P)$ similarly. If the supremum (resp. infimum) of $f$ is undefined on an interval, we say that the upper sum (resp. lower sum) is $\infty$ (resp. $-\infty$).

**Lemma 1.44.** *For any bounded real valued functions $f$ and $g$ on an interval $[a, b]$ the following hold:*

(I.) $U(f + g, P) \le U(f, P) + U(g, P)$

(II.) $L(f + g, P) \ge L(f, P) + L(g, P)$

(III.) $-U(f, P) = L(-f, P)$

(IV.) $-L(f, P) = U(-f, P)$

*Proof.* We only prove (I.) and (III.), as (II.) and (IV.) are very similar.

(I.) $U(f + g, P) = \sum_{i=1}^{n} M_i \Delta x_i$ where $M_i = \sup(f + g)(x)$, $x \in [x_{i-1}, x_i]$. Then $M_i \le \sup f(x) + \sup g(x) = M_{f,i} + M_{g,i}$ and it follows that $\sum_{i=1}^{n} M_i \Delta x_i \le \sum_{i=1}^{n} (M_{f,i} + M_{g,i}) \Delta x_i = \sum_{i=1}^{n} M_{f,i} \Delta x_i + \sum_{i=1}^{n} M_{g,i} \Delta x_i = U(f, P) + U(g, P)$.

(III.) $-U(f, P) = -\sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} -M_i \Delta x_i$ and $-M_i = -\sup f(x) = \inf -f(x)$ and the result follows. $\square$

**Lemma 1.45.** *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let $B_{(f,P)} = \{[L(f, P), U(f, P)]\}$ and let $\mathcal{B}_f = \{B_{(f,P)} : P$ is a partition of $I\}$ Then $\mathcal{B}_f$ is a filter base.*

*Proof.* Since the partition $\{a, b\}$ is always valid and the upper and lower sums are always defined it is clear that $\mathcal{B} \ne \varnothing$. We verify axioms (B. 1-2).

(B. 1) If $B \in \mathcal{B}$ then $B = \{[L(f, P), U(f, P)]\}$ for some partition $P$. This is nonvoid for all paritions since the upper and lower sums are always defined, thus $B \ne \varnothing$.

(B. 2) Let $B_{(f,P)}, B_{(f,Q)} \in \mathcal{B}$ then there is a common refinement of the paritions $P$ and $Q$ (i.e. taking all the points of $P$ and $Q$ and reordering them to form a new parition) say $R$. Then the set $B_{(f,R)} \subseteq B_{(f,P)} \cap B_{(f,Q)}$ and since $R$ is a partition, $B_{(f,R)} \in \mathcal{B}_f$.

$\square$

**Definition 1.46.** We say a bounded function $f : [a, b] \to \mathbb{R}$ is **Riemann Integrable** if the filter generated by $\mathcal{B}_f$ is convergent. Moreover, the value of the limit is called the **definite integral** of $f$ from $a$ to $b$ and is denoted $\lim \mathcal{B}_f = \int_a^b f$.

**Theorem 1.47.** *Suppse $f$ and $g$ are Riemann Integrable functions, then:*

(I.) *$f + g$ is Riemann Integrable*

(II.) *$-f$ is Riemann Integrable*

(III.) *$fg$ is Riemann Integrable*

(IV.) *If $g(x) \neq 0 \ \forall x \in I$ then $f/g$ is Riemann Integrable*

*Proof.* (I.) Using Lemma 1.44 we see that for any partition $P$, $B_{f+g,P} \subset [L(f,P) + L(g,P), U(f,P) + U(g,P)] = B_{(f,P)} + B_{(g,P)}$ where addition is taken as the usual pointwise addition of sets. Let $x = \lim \mathcal{B}_f + \lim \mathcal{B}_g = x_f + x_g$ and we claim $\mathcal{B}_{f+g}$ converges to $x$. We know that $\mathcal{B}_f \geq \mathcal{N}(x_f)$ and $\mathcal{B}_g \geq \mathcal{N}(x_g)$ so $\mathcal{B}_{f+g} \geq \mathcal{B}_f + \mathcal{B}_g \geq \mathcal{N}(x_f) + \mathcal{N}(x_g) = \mathcal{N}(x)$ and so $\mathcal{B}_{f+g}$ converges to $x$.

(II.) Applying Lemma 1.44 yields that $\mathcal{B}_{-f}$ converges to $-\lim \mathcal{B}_f$ and so $-f$ is Riemann Integrable

(III.) Using a similar argument to (I.) one can show that if $\lim \mathcal{B}_f = x_f$ and $\lim \mathcal{B}_g = x_g$ then $fg$ converges to $x_f x_g$

(IV.) Use (III.) since $1/g$ is defined everywhere on the (closed) interval. $\qquad\square$

**Theorem 1.48.** *Let $f : [a, b] \to \mathbb{R}$ be Riemann Integrable. Let $c \in [a, b]$ then $\int_a^b f = \int_a^c f + \int_c^b f$*

*Proof.* We note that the set of all partitions of $[a, c] \cup [c, b]$ is a subset of all paritions of $[a, b]$. Hence the filter generated by the filter base $\mathcal{C}_f = \{B_{(f,P)} : P \text{ is a parition of } [a, c] \cup [c, b]\}$ is finer than $\mathcal{B}_f$ and hence convergent (to the same limit). That is, the filter base $\mathcal{C}_f = \{B_{(f,P)} : P \text{ is a parition of } [a, c]\} + \{B_{(f,P)} : P \text{ is a parition of } [c, b]\}$ is convergent. It follows that $\int_a^b f = \int_a^c f + \int_c^b f$. $\quad\square$

**Remark 1.49.** I have silently used here the fact that if $\mathcal{F}$ and $\mathcal{G}$ are filters over the same set $X$, both convergent to say $x_{\mathcal{F}}$ and $x_{\mathcal{G}}$ respectively, then $\mathcal{F} + \mathcal{G}$ converges to $x_{\mathcal{F}} + x_{\mathcal{G}}$. I havenn't actually included a proof of this, however it probably should be included! Let me know what you think.

**Theorem 1.50.** *(Mean Value Theorem for Integrals) Let $f : [a, b] \to \mathbb{R}$ be Riemann Integrable and continuous. Then there exists an $\xi \in [a, b]$ such that $\int_a^b f = f(\xi)(b - a)$.*

*Proof.* Since $[a, b]$ is compact we know that f attains a minimum and maximum value on $[a, b]$ say $m$ and $M$ respectively. Note that

$$(b-a)m \leq L(f, P) \leq U(f, P) \leq (b-a)M$$

For any partition $P$ of $[a, b]$. Then since $\mathcal{B}_f$ is convergent to say $x$ we have that

$$m \leq \frac{L(f, P)}{(b-a)} \leq \frac{x}{(b-a)} \leq \frac{U(f, P)}{(b-a)} \leq M$$

By the intermediate value theorem we know that $f$ takes all values between $m$ and $M$ so there exists an $\xi \in [a, b]$ such that $f(\xi) = \frac{x}{(b-a)}$ or equivalently: $(b-a)f(\xi) = \int_a^b f$. $\qquad\square$

**Remark 1.51.** The same approach can be used to prove the generlized Mean Value Theorem for Integrals which states that if $f$ is Riemann Integrable and continuous and $\phi$ is any integrable function then there is an $\xi \in [a, b]$ such that $\int_a^b f(t)\phi(t)dt = f(\xi)\int_a^b \phi(t)dt$

**Remark 1.52.** One can use the Mean Value Theorem for Integrals to prove the First Fundamental Theorem of Calculus by using a standard argument. The second Fundamental Theorem of Calculus can be proven in the following way:

**Theorem 1.53.** *(Second Fundamental Theorem of Calculus) If $f$ is Riemann Integrable and $g$ is any antiderivative of $f$ then $\int_a^b f = g(b) - g(a)$.*

*Proof.* Let $P$ be any partition of $[a, b]$. Then $g(b) - g(a) = \sum_{i=1}^n g(x_i) - g(x_i)$ Since $g$ is differentiable (to $f$) we can invoke the mean value theorem on each interval $[x_{i-1}, x_i]$ to obtain:

$$\sum_{i=1}^n g(x_i) - g(x_i) = \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}), \text{ for some } \xi_i \in (x_{i-1}, x_i)$$

Since $\inf f(x) \leq f(\xi_i) \leq \sup f(x)$ for $x \in [x_{i-1}, x_i]$ we have that:

$$L(f, P) \leq \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) \leq U(f, P)$$

$$\implies g(b) - g(a) = \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) \in B_{(f, P)}$$

Since this is valid for all partitions $P$, it follows that $\lim \mathcal{B}_f = g(b) - g(a)$. $\quad\square$

To close the section, we prove that the definiton of the Riemann Integral here is equivalent to the usual definition of the Riemann Integral. We have defered this to now so we could demonstrate how the filter definition can be used to prove the same facts about the integral, without simply appealing to the more standardized proofs. We will assume for brevity that the reader is already familiar with tagged partitions and meshs of partitions.

**Theorem 1.54.** *The Riemann Integral defined in Definition 1.46 is equivalent to the following definition:*
$\int_a^b f = s \iff$ *for all $\epsilon > 0$ there exists a $\delta > 0$ and a tagged partition $P$ whose mesh is less than $\delta$ satisfying $\left| \sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i) - s \right| < \epsilon$ ($\star$)*

*Proof.* It will be more convenient for us to rewrite ($\star$) in the following form:

$$s - \epsilon < \sum_{i=1}^n f(t_i)(x_{i+1} - x_i) < s + \epsilon$$

($\Rightarrow$) Suppose $\mathcal{B}_f$ is convergent to $s$, then for all $\epsilon > 0$ there exists a partition $P$ such that $B_{(f,P)} \subseteq (s - \epsilon, s + \epsilon)$. In particular we can set $\delta$ equal to the width of the largest interval as required in ($\star$), moreover this implies:

$$s - \epsilon < L(f, P) < U(f, P) < s + \epsilon$$

Since $\inf f(x) \le f(t_i) \le \sup f(x)$ on each $[x_{i-1}, x_i]$ it follows that:

$$s - \epsilon < L(f, P) < \sum_{i=1}^n f(t_i)(x_{i+1} - x_i) < U(f, P) < s + \epsilon$$

i.e. ($\star$) holds.
($\Leftarrow$) Suppose now that $\star$ holds and let $P$ be a parition of $[a, b]$ satisfying $|x_i - x_{i-1}| < \delta \; \forall i$. We note that ($\star$) must hold for any choice of tagging of the intervals $[x_{i-1}, x_i]$, in particular for when $f(t_i) = \sup f(x)$ for $x \in [x_{i-1}, x_i]$ and when $f(t_i) = \inf f(x)$ for $x \in [x_{i-1}, x_i]$. Thus it follows that:

$$s - \epsilon < L(f, P) < U(f, P) < s + \epsilon$$

For all partitions finer than $P$. This means any element of $N(x) \in \mathcal{N}(x)$ we can find a $B_{(f,P)} \subseteq N(x)$ using the result above and hence $\mathcal{B}_f$ is convergent. $\quad \square$

## 1.4 Applications of Filters to Compactness

Perhaps the most interesting application of filters is to that of Compactness. Our goal in this section is to work towards a proof of Tychonoff's Theorem on the arbitrary product of compact subsets following the approach of Gaal. To do so we will need to define what the product of two (or more) filters is and prove a number of Lemmas. In doing so we will obtain a number of results which will be used to characterise compactness in a more general setting. To begin, let us note that in a metric space a subset is compact if and only if it satisfies the Bolzano-Weierstra property - we can generalise this to an arbitrary topological space using filters as follows:

**Theorem 1.55.** *A topological space $X$ is compact if and only if every filter $\mathcal{F}$ in $X$ has a non-void adherence.*

*Proof.* Necessity follows from the fact that a every closed family of sets of a compact set having the finite intersection property has a non-empty intersection. $\overline{\mathcal{F}}$ satisfies the finite intersection property by (F. 3) and moreover is a familiy of closed sets, hence $\bigcap_{F \in \mathcal{F}} \overline{F}$ is nonvoid.

To prove sufficiency we proceed by contraposition. Suppose that $X$ is not compact, then there exists a family of closed subsets of $X$, $\mathcal{B}$, having the finite intersection property but whose total intersection is void. Then $\mathcal{B}$ is a filter base which generates a filter on $X$ which has void adherence. $\qquad \square$

This Theorem has an important extension to ultrafilters:

**Theorem 1.56.** *A topological space $X$ is compact if and only is every ultrafilter in $X$ in convergent*

*Proof.* ($\Rightarrow$) By Theorem 1.55 every Filter in $X$ has a non-void adherence and so by Theorem 1.37 an ultrafilter $\mathfrak{U}$ on $X$ must be convergent.

($\Leftarrow$) Suppose every ultrafilter on $X$ is convergent. Then by Theorem 1.14 every filter $\mathcal{F}$ in X has an ultrafilter $\mathfrak{U}$ finer than $\mathcal{F}$ and so by Lemma 1.35, $\lim \mathfrak{U} \subseteq$ adh $\mathcal{F} \neq \varnothing$. $\qquad \square$

This characterisation of compactness is so useful that it will become our definition of compactness when we move to a convergence setting. We now turn to some definitions and lemmas concerning the products of filters:

**Definition 1.57.** Let $X = \prod X_s$ be a product set and let $\mathcal{F}_s$ be a filter in each of the factors $X_s$. Then the set of all products $\prod F_s$ where $F_s \in \mathcal{F}_s$ and $F_s = X_s$ for all but finitely many indices is a filter base $\mathcal{B}$ in $X$. The filter generated by this base is called the **product** of the filters $\mathcal{F}_s$ and will be denoted $\prod \mathcal{F}_s$. It is the coarsets filter in $X$ such that it's projection in $X_s$ is $\mathcal{F}_s$.

**Lemma 1.58.** *Let $\mathcal{F}$ be a filter in a the product $X = \prod X_s$. Then the family $\mathcal{F}_s = \{F_s\}$ where $F_s$ denotes the projection of $F \in \mathcal{F}$ onto $X_s$ is a filter in $X_s$ (called the projection of $\mathcal{F}$ into $X_s$)*

*Proof.* We verify the axioms of a filter:

(F. 1) $F_s = \varnothing \iff F = \varnothing$ which can't be true since $\mathcal{F}$ is a filter

(F. 2) Note that for $F_1, F_2 \in \mathcal{F}$, there is an $F_3 \in \mathcal{F}$ such that $F_1 \cap F_2 \supseteq F_3$ which implies that $F_{1,s} \cap F_{2,s} \supseteq F_{3,s}$.

(F. 3) Let $A_s$ be set in $X_s$ containing $F_s$ of a set $F \in \mathcal{F}$. Then the set $A = \{x : x_s \in A_s\}$ contains $F$ and so it is in $\mathcal{F}$ since it is a filter. Hence, $A_s$ itself is a projection of a set $A \in \mathcal{F}$ so that $F_s \subseteq A_s \in \mathcal{F}_s$.

$\qquad \square$

**Lemma 1.59.** *If $\mathcal{F} \leq \mathcal{G}$ then $\mathcal{F}_s \leq G_s$ for every index s.*

*Proof.* If $F_s \in \mathcal{F}_s$ is the projection of $F \in \mathcal{F}$ then $F \in \mathcal{G}$ and so $F_s \in \mathcal{G}_s$. Hence $\mathcal{F}_s \leq \mathcal{G}_s$. $\qquad \square$

**Lemma 1.60.** *If $\mathfrak{U}$ is an ultrafilter in the product set $X$, then the projection of $\mathfrak{U}$ into each of the factors $X_s$ is an ultrafilter*

*Proof.* Let $\mathcal{F}$ be a filter in $X = \prod X_s$ and let $\mathcal{F}_s$ denote the projection as usual. Suppose that we can find some filter $G_s$ in $X_s$ which is strictly finer than $F_s$. Then there is a set $G_s$ in $X_s$ such that $F_s \not\subseteq G_s$ for every $F_s \in \mathcal{F}_s$. We introduce the nonvoid sets $\phi(F) = \{x : x \in F$ and $x_s \in G_s\}$. Then the family $\mathcal{B} = \{\phi(F) : F \in \mathcal{F}\}$ is a filter base in $X$ which generates a filter which is not coarser than $\mathcal{F}$ because $\phi(F) \subseteq F$ for every $F \in \mathcal{F}$. Moreover, if $F \in \mathcal{F}$, then $\phi(F_s) \subseteq \mathcal{G}_s$ while $F_s \not\subseteq G_s$ so $\mathcal{B}$ generates a filter strictly finer than $\mathcal{F}$ and so $\mathcal{F}$ is not an ultrafilter. This completes the proof. $\square$

**Example 1.61.** The converse to the previous Lemma does not hold, i.e. we can find a filter $\mathcal{F}$ in $X$ such that its projection into every $X_s$ is an ultrafilter, but $\mathcal{F}$ is not. Let $\mathcal{F}$ be the product of a family of ultrafilters $\mathfrak{U}_s$ in the factors $X_s$ which each contain a fixed point $x_s$. Then by our construction of the product filter, the point $x \in X$ whose $s$th coordinate is $x_s$ is not in $\mathcal{F}$ so there exists a filter $\mathcal{G} > \mathcal{F}$ containing $x$ so $\mathcal{F}$ cannot be an ultrafilter, yet it's projection onto each $X_s$ is.

**Lemma 1.62.** *If $\mathcal{F}$ is a filter in a product space $X$, then $(adh\ \mathcal{F})_s \subseteq adh\ F_s$ and $(\lim \mathcal{F})_s \subseteq \lim \mathcal{F}_s$ for every index $s$.*

*Proof.* Suppose that $x \in$ adh $\mathcal{F}$ s that $x_s \in (\text{adh } \mathcal{F})_s$. If $N_{x_s}$ is a neighbourhood of $x_s$ in $X_s$, then $N_x = \{\xi : \xi \in X, \xi_s \in N_{x_s}\}$ is a neighbourhood of $x \in X$. Since $x$ is an adherence point of $\mathcal{F}$ we have $N_x \cap F \neq \varnothing$ for every $F \in \mathcal{F}$ and so $N_{x_s} \cap \mathcal{F}_s \neq \varnothing$ for every $F_s \in \mathcal{F}_s$. Thus $x_s \in$ adh $\mathcal{F}_s$ and $(\text{adh } \mathcal{F})_s \subseteq$ adh $\mathcal{F}_s$. Similarly, if $x \in \lim \mathcal{F}$ then there is an $F \in \mathcal{F}$ such that $F \subseteq N_x$. Therefore $F_s \subseteq N_{x_s}$ and this shows that $x_s \in \lim F_s$. $\square$

Unlike Lemma 1.60, the second part of Lemma 1.62 has a converse. Namely:

**Lemma 1.63.** *A filter $\mathcal{F}$ in $X$ is convergent if and only if every projection $\mathcal{F}_s$ in $X_s$ is convergent.*

*Proof.* ($\Rightarrow$) Follows from Lemma 1.62
($\Leftarrow$) For each index $s$ we choose a point $x_s$ in $\lim \mathcal{F}_s$. Let $x \in X$ be the point whose $s$th coordinate is $x_s$. We show that $x \in \lim \mathcal{F}$. Let $O = \prod O_s$ be an open set in $X$ which contains $x$. By the definition of the product topology, $O_s = X_s$ for all but finitely many indices. For each $s$ there is an $F^s \in \mathcal{F}$ such that $F^s_s \subseteq O_s$ because $x_s \in O_s$ and each $x_s \in \lim \mathcal{F}_s$. We may choose $F^s = X$ for all but finitely many indices. Then $F = \bigcap F^s$ is a finite intersection and so $F \in \mathcal{F}$. Moreover, $F \subseteq \prod F^s_s \subseteq \prod O_s = O$. Hence $x \in \lim \mathcal{F}$. $\square$

**Remark 1.64.** We have actually proven something slightly stronger, namely: $x \in \lim \mathcal{F}$ provided $x_s \in \lim F_s$ for every $s$ and hence $\lim \mathcal{F} \supseteq \prod \lim \mathcal{F}_s$. Combining this with the rest of Lemma 1.62 we obtain the following result:

**Lemma 1.65.** *If $\mathcal{F}$ is a filter in a product space then $\lim \mathcal{F} = \prod \lim \mathcal{F}_s$.*

**Example 1.66.** Interestingly, the first half of Lemma 1.62 does not have a converse even though the second half does. We can construct a counter example as follows: Let $X_1 = X_2$ be the set of reals under the discrete topology. We consider the family $\mathcal{F}$ of all sets $F$ in $X = X_1 \times X_2$ having the property that the lines $\{(\xi_1, x_2) : \xi_1 \in X_1\}$ and $\{(x_1, \xi_2) : \xi_2 \in X_2\}$ are subsets of $F$ for all but finitely many values of $x_1 \in X_1$ and $x_2 \in X_2$. Then $\mathcal{F}$ is clearly a filter and it's projections are $\mathcal{F}_1 = X_1$ and $\mathcal{F}_2 = X_2$ so adh $\mathcal{F}_1 = X_1$ and adh $\mathcal{F}_2 = X_2$. However, the filter $\mathcal{F}$ does not have any adherence points because the set $\{(x_1, x_2)\}$ consisting of the single point $(x_1, x_2)$ is neighbourhood of $(x_1, x_2)$ and its complement is an element of $\mathcal{F}$. This counter example also demonstrates that in general, adh $\mathcal{F}$ and $\prod$ adh $\mathcal{F}$ are different sets.

We now are in a position to prove Tychonoff's Theorem.

**Theorem 1.67.** *(Tychonoff's Theorem) The product of any collection of compact topological spaces is compact.*

*Proof.* Let $\mathfrak{U}$ be an ultrafilter in the product space $X = \prod X_s$ where each $X_s$ is compact. Then by Lemma 1.60 the projection $\mathfrak{U}_s$ is an ultrafilter in $X_s$ and so by Theorem 1.56 it is convergent. Then Lemma 1.63 implies that $\mathfrak{U}$ is convergent. Thus every ultrafilter $\mathfrak{U}$ in $X$ is convergent and so by Theorem 1.56 $X$ is compact. $\qquad\square$

**Remark 1.68.** It is well known that Tychonoff's Theorem is equivalent to the Axiom of Choice. In this proof we have invoked the Axiom of Choice through using Zorn's Lemma to postulate the existence of ultrafilters which we needed in Theorem 1.56.