Formula Sheet

- Conservation of probability

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}J(x,t) = 0$$

$$\rho(x,t) = |\psi(x,t)|^2 \; ; \qquad J(x,t) = \frac{\hbar}{2im}\left[\psi^* \frac{\partial}{\partial x}\psi - \psi \frac{\partial}{\partial x}\psi^*\right]$$

- Variational principle:

$$E_{gs} \leq \int dx\, \psi^*(x) H \psi(x)\,, \quad \text{for all } \psi(x) \text{ satisfying } \int dx \psi^*(x)\psi(x) = 1$$

- Spin-1/2 particle:

$$\text{Stern-Gerlach :} \quad H = -\vec{\mu}\cdot\vec{B}\,, \qquad \vec{\mu} = g\;\frac{e\hbar}{2m}\;\frac{1}{\hbar}\vec{S} = \gamma\vec{S}$$

$$\mu_{B} = \frac{e\hbar}{2m_e}\,, \qquad \vec{\mu}_e = -2\,\mu_{B}\frac{\vec{S}}{\hbar}\,,$$

$$\text{In the basis } |1\rangle \equiv |z;+\rangle = |+\rangle = \begin{pmatrix}1\\0\end{pmatrix}\,,\; |2\rangle \equiv |z;-\rangle = |-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$S_i = \frac{\hbar}{2}\sigma_i \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\;;\; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\;;\; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \quad \rightarrow \quad \big[S_i\,,\,S_j\big] = i\hbar\,\epsilon_{ijk}S_k \quad (\epsilon_{123} = +1)$$

$$\sigma_i\sigma_j = \delta_{ij}I + i\epsilon_{ijk}\sigma_k \quad \rightarrow \quad (\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b}\,I + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

$$e^{i\mathbf{M}\theta} = \mathbf{1}\cos\theta + i\mathbf{M}\sin\theta\,, \quad \text{if } \mathbf{M}^2 = \mathbf{1}$$

$$\exp\big(i\vec{a}\cdot\vec{\sigma}\big) = \mathbf{1}\cos a + i\vec{\sigma}\cdot\left(\frac{\vec{a}}{a}\right)\sin a\,, \quad a = |\vec{a}|$$

$$\begin{aligned} \exp(i\theta\sigma_3)\,\sigma_1\,\exp(-i\theta\sigma_3) &= \sigma_1\cos(2\theta) - \sigma_2\sin(2\theta) \\ \exp(i\theta\sigma_3)\,\sigma_2\,\exp(-i\theta\sigma_3) &= \sigma_2\cos(2\theta) + \sigma_1\sin(2\theta)\;. \end{aligned}$$

$$S_{\vec{n}} = \vec{n}\cdot\vec{S} = n_xS_x + n_yS_y + n_zS_z = \frac{\hbar}{2}\,\vec{n}\cdot\vec{\sigma}\;.$$

$$(n_x, n_y, n_z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,, \quad S_{\vec{n}}\,|\vec{n};\pm\rangle = \pm\frac{\hbar}{2}\,|\vec{n};\pm\rangle$$

$$\begin{aligned} |\vec{n};+\rangle &= \cos(\theta/2)|+\rangle + \sin(\theta/2)\exp(i\phi)|-\rangle \\ |\vec{n};-\rangle &= -\sin(\theta/2)\exp(-i\phi)|+\rangle + \cos(\theta/2)|-\rangle \end{aligned}$$

- Complete orthonormal basis $|i\rangle$

$$\langle i|j\rangle = \delta_{ij}\,, \qquad \mathbf{1} = \sum_i |i\rangle\langle i|$$

$$\mathcal{O}_{ij} = \langle i|\mathcal{O}|j\rangle \quad \leftrightarrow \quad \mathcal{O} = \sum_{i,j} \mathcal{O}_{ij}\ |i\rangle\langle j|$$

$$\langle i|A^\dagger|j\rangle = \langle j|A|i\rangle^*$$

$$\text{hermitian operator: } \mathcal{O}^\dagger = \mathcal{O}, \quad \text{unitary operator: } U^\dagger = U^{-1}$$

- Position and momentum representations: $\psi(x) = \langle x|\psi\rangle$; $\tilde{\psi}(p) = \langle p|\psi\rangle$;

$$\hat{x}|x\rangle = x|x\rangle\,, \quad \langle x|y\rangle = \delta(x-y)\,, \quad \mathbf{1} = \int dx\ |x\rangle\langle x|\,, \quad \hat{x}^\dagger = \hat{x}$$

$$\hat{p}|p\rangle = p|p\rangle\,, \quad \langle q|p\rangle = \delta(q-p)\,, \quad \mathbf{1} = \int dp\ |p\rangle\langle p|\,, \quad \hat{p}^\dagger = \hat{p}$$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\Big(\frac{ipx}{\hbar}\Big)\,; \qquad \tilde{\psi}(p) = \int dx \langle p|x\rangle\langle x|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int dx \exp\Big(-\frac{ipx}{\hbar}\Big)\psi(x)$$

$$\langle x|\hat{p}^n|\psi\rangle = \Big(\frac{\hbar}{i}\frac{d}{dx}\Big)^n \psi(x)\,; \qquad \langle p|\hat{x}^n|\psi\rangle = \Big(i\hbar\frac{d}{dp}\Big)^n \tilde{\psi}(p)\,; \qquad [\hat{p}, f(\hat{x})] = \frac{\hbar}{i} f'(\hat{x})$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(ikx) dx = \delta(k)$$

- Generalized uncertainty principle

$$(\Delta A)^2 \equiv \langle (A - \langle A\rangle)^2\rangle = \langle A^2\rangle - \langle A\rangle^2$$

$$(\Delta A)^2(\Delta B)^2 \geq \Big(\langle\Psi|\frac{1}{2i}[A,B]|\Psi\rangle\Big)^2$$

$$\Delta x\,\Delta p \geq \frac{\hbar}{2}$$

$$\Delta x = \frac{\Delta}{\sqrt{2}} \ \text{ and } \ \Delta p = \frac{\hbar}{\sqrt{2}\Delta} \ \text{ for a gaussian wavefuntion } \psi \sim \exp\Big(-\frac{1}{2}\frac{x^2}{\Delta^2}\Big)$$

$$\int_{-\infty}^{+\infty} dx \exp\left(-ax^2\right) = \sqrt{\frac{\pi}{a}}$$

$$\Delta H \Delta t \geq \frac{\hbar}{2}\,, \quad \Delta t \equiv \frac{\Delta Q}{\big|\frac{d\langle Q\rangle}{dt}\big|}$$

- Commutator identities

$$[A,BC] = [A,B]C + B[A,C]\,,$$
$$e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \ldots\,,$$
$$e^A B e^{-A} = B + [A,B]\,, \quad \text{if} \quad [[A,B],A] = 0\,,$$
$$[\,B\,,\, e^A\,] = [\,B\,,A\,]e^A\,, \qquad \text{if} \quad [[A,B],A] = 0$$
$$e^{A+B} \;=\; e^A e^B e^{-\frac{1}{2}[A,B]} \;=\; e^B e^A e^{\frac{1}{2}[A,B]}\;, \quad \text{if } [A,B] \text{ commutes with } A \text{ and with } B$$

- Time evolution

$$|\Psi,t\rangle = U(t,0)|\Psi,0\rangle\,, \quad U \;\; \text{unitary}$$

$$U(t,t) = \mathbf{1}\,, \quad U(t_2,t_1)U(t_1,t_0) = U(t_2,t_0)\,, \quad U(t_1,t_2) = U^\dagger(t_2,t_1)$$

$$i\hbar\frac{d}{dt}|\Psi,t\rangle = \hat{H}(t)|\Psi,t\rangle \quad \leftrightarrow \quad i\hbar\frac{d}{dt}U(t,t_0) = \hat{H}(t)U(t,t_0)$$

$$\text{Time independent } \hat{H}\text{:} \quad U(t,t_0) = \exp\Big[-\frac{i}{\hbar}\,\hat{H}(t-t_0)\Big] = \sum_n e^{-\frac{i}{\hbar}\,E_n(t-t_0)}|n\rangle\langle n|$$

$$[\,\hat{H}(t_1)\,,\hat{H}(t_2)\,] = 0, \quad \forall\, t_1,t_2, \quad U(t,t_0) = \exp\Big[-\frac{i}{\hbar}\int_{t_0}^{t_1} dt'\hat{H}(t')\Big]$$

$$\langle A\rangle = \langle\Psi,t|A_S|\Psi,t\rangle = \langle\Psi,0|A_H(t)|\Psi,0\rangle \quad \to \quad A_H(t) = U^\dagger(t,0)A_S U(t,0)$$

$$[A_S,\,B_S\,] = C_S \quad \to \quad [A_H(t)\,,\,B_H(t)\,] = C_H(t)$$
$$i\hbar\frac{d}{dt}\hat{A}_H(t) = [\hat{A}_H(t),\hat{H}_H(t)]\,, \quad \text{for } A_S \text{ time-independent}$$

- Two state systems

$$H = -\gamma\,\vec{S}\cdot\vec{B} \quad \to \quad \text{spin vector } \vec{n} \text{ precesses with Larmor frequency } \vec{\omega} = -\gamma\vec{B}$$
$$\text{NMR magnetic field } \vec{B} = B_0\vec{e}_z + B_1\big(\cos\omega t\,\vec{e}_x - \sin\omega t\,\vec{e}_y\big)$$
$$|\psi(t)\rangle = \exp\Big(\frac{i}{\hbar}\omega t S_z\Big)\exp\Big(\frac{i}{\hbar}\gamma\vec{B}_{\text{eff}}\cdot\vec{S}\,t\Big)|\psi(0)\rangle$$
$$\vec{B}_{\text{eff}} = B_0\big(1-\frac{\omega}{\omega_0}\big)\vec{e}_z + B_1\,\vec{e}_x$$

- Harmonic Oscillator

$$\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)\,, \quad \hat{N} = \hat{a}^\dagger\hat{a}$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\,, \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)\,,$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)\,, \qquad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})\,,$$

$$[\hat{x}, \hat{p}] = i\hbar\,, \quad [\hat{a}, \hat{a}^\dagger] = 1\ .$$

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$$

$$\hat{H}|n\rangle = E_n|n\rangle = \hbar\omega\big(n + \frac{1}{2}\big)|n\rangle\,, \quad \hat{N}|n\rangle = n|n\rangle\,, \quad \langle m|n\rangle = \delta_{mn}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\,, \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle\ .$$

$$\psi_0(x) = \langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\ .$$

$$x_H(t) = \hat{x}\,\cos\omega t + \frac{\hat{p}}{m\omega}\sin\omega t$$

$$p_H(t) = \hat{p}\,\cos\omega t - m\omega\,\hat{x}\sin\omega t$$

- Coherent states and squeezed states

$$T_{x_0} \;\equiv\; e^{-\frac{i}{\hbar}\hat{p}\,x_0}\,, \quad T_{x_0}|x\rangle = |x + x_0\rangle$$

$$|\tilde{x}_0\rangle \;\equiv\; T_{x_0}|0\rangle \;= e^{-\frac{i}{\hbar}\hat{p}x_0}|0\rangle\,,$$

$$|\tilde{x}_0\rangle = e^{-\frac{1}{4}\frac{x_0^2}{d^2}}e^{\frac{x_0}{\sqrt{2}d}a^\dagger}|0\rangle\,, \quad \langle x|x_0\rangle = \psi_0(x - x_0)\,, \quad d^2 = \frac{\hbar}{m\omega}$$

$$|\alpha\rangle \equiv D(\alpha)|0\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle\,, \quad D(\alpha) \equiv \exp\Big(\alpha a^\dagger - \alpha^* a\Big)\,, \quad \alpha = \frac{\langle\hat{x}\rangle}{\sqrt{2}\,d} \;+\; i\,\frac{\langle\hat{p}\rangle\,d}{\sqrt{2}\,\hbar}\,.$$

$$|\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\,a^\dagger}|0\rangle\,, \quad \hat{a}|\alpha\rangle = \alpha|\alpha\rangle\,, \quad |\alpha, t\rangle \;=\; e^{-i\omega t/2}|e^{-i\omega t}\alpha\rangle$$

$$|0_\gamma\rangle \;=\; S(\gamma)|0\rangle\,, \quad \text{with} \quad S(\gamma) \;=\; \exp\Big(-\frac{\gamma}{2}(a^\dagger a^\dagger - aa)\Big)$$

$$|0_\gamma\rangle \;=\; \frac{1}{\sqrt{\cosh\gamma}}\exp\Big(-\frac{1}{2}\,\tanh\gamma\,a^\dagger a^\dagger\Big)\,|0\rangle$$

$$S^\dagger(\gamma)\,aS(\gamma) = \cosh\gamma\,a - \sinh\gamma\,a^\dagger\,, \quad D^\dagger(\alpha)\,aD(\alpha) = a + \alpha$$

$$|\alpha, \gamma\rangle \equiv D(\alpha)S(\gamma)|0\rangle$$

- Orbital angular momentum operators

$$\hat{L}_i = \epsilon_{ijk}\, \hat{x}_j\, \hat{p}_k$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial \theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}\ ; \qquad \hat{L}_\pm = \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

- Spherical Harmonics

$$Y_{\ell,m}(\theta,\phi) \equiv \langle\theta,\phi|\ell,m\rangle$$

$$Y_{0,0}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}\ ; \qquad Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\exp(\pm i\phi)\ ; \qquad Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{2,\pm2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta\exp(\pm 2i\phi)\ ; \qquad Y_{2,\pm1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\exp(\pm i\phi)\ ;$$

$$Y_{2,0}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right)$$

- Algebra of angular momentum operators $\vec{J}$ (orbital or spin, or sum)

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k\ ; \quad [\,J^2,\ J_i] = 0$$

$$J^2|jm\rangle = \hbar^2\, j(j+1)|jm\rangle\ ; \qquad J_z|jm\rangle = \hbar m|jm\rangle\,, \quad m = -j, \ldots, j\,.$$

$$J_\pm = J_x \pm iJ_y\,, \quad (J_\pm)^\dagger = J_\mp \quad J_x = \frac{1}{2}(J_+ + J_-)\,,\ J_y = \frac{1}{2i}(J_+ - J_-)$$

$$[J_z\,, J_\pm] = \pm\hbar\, J_\pm\,, \quad [J^2, J_\pm] = 0\ ; \qquad [J_+, J_-] = 2\hbar\, J_z$$

$$J^2 = J_+J_- + J_z^2 - \hbar J_z = J_-J_+ + J_z^2 + \hbar J_z$$

$$J_\pm|jm\rangle = \hbar\sqrt{j(j+1) - m(m\pm1)}\ |j, m\pm1\rangle$$

- Radial equation

$$\Big(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}+V(r)+\frac{\hbar^2\,\ell(\ell+1)}{2mr^2}\Big)u_{\nu\ell}(r)=E_{\nu\ell}u_{\nu\ell}(r)\quad\text{(bound states)}$$

$$u_{\nu\ell}(r)\sim r^{\ell+1}\,,\quad\text{as } r\to 0\,.$$

$$E>0 \text{ and } \lim_{r\to\infty}V(r)=0\,,\quad \lim_{r\to\infty}u_\ell(r)=\sin\Big(kr-\ell\,\frac{\pi}{2}+\delta_\ell(E)\Big)\,,\quad k=\frac{\sqrt{2mE}}{\hbar}$$

- Hydrogen atom

$$E_n=-\frac{e^2}{2a_0}\frac{1}{n^2}\,,\qquad \psi_{n,\ell,m}(\vec{x})=\frac{u_{n\ell}(r)}{r}Y_{\ell,m}(\theta,\phi)$$

$$n=1,2,\ldots\,,\qquad \ell=0,1,\ldots,n-1\,,\quad m=-\ell,\ldots,\ell$$

$$a_0=\frac{\hbar^2}{me^2}\,,\qquad \alpha=\frac{e^2}{\hbar c}\simeq\frac{1}{137}\,,\qquad \hbar c\simeq 200\,\text{MeV–fm}$$

$$u_{1,0}(r)=\frac{2r}{a_0^{3/2}}\exp(-r/a_0)$$

$$u_{2,0}(r)=\frac{2r}{(2a_0)^{3/2}}\left(1-\frac{r}{2a_0}\right)\exp(-r/2a_0)$$

$$u_{2,1}(r)=\frac{1}{\sqrt{3}}\frac{1}{(2a_0)^{3/2}}\frac{r^2}{a_0}\exp(-r/2a_0)$$

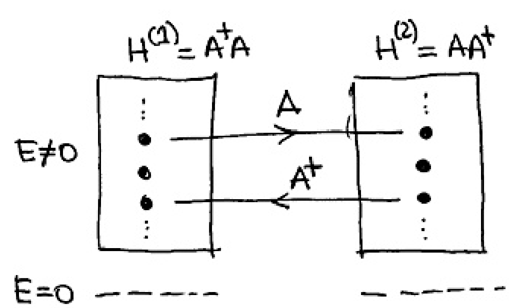
- Factorization method

$$A=\frac{d}{dx}+W(x)\,,\quad A^\dagger=-\frac{d}{dx}+W(x)$$

$$H^{(1)}=A^\dagger A=-\frac{d^2}{dx^2}+W^2-W'$$

$$H^{(2)}=AA^\dagger=-\frac{d^2}{dx^2}+W^2+W'$$

$$\phi_0^{(1)}=N\exp\Big(-\int^x W(x')dx'\Big)$$



- Addition of Angular Momentum $\vec{J} = \vec{J}_1 + \vec{J}_2$

$$\text{Uncoupled basis :} \quad |j_1 j_2; m_1 m_2\rangle \quad \text{CSCO} : \{J_1^2, J_2^2, J_{1z}, J_{2z}\}$$

$$\text{Coupled basis :} \quad |j_1 j_2; jm\rangle \quad \text{CSCO} : \{J_1^2, J_2^2, J^2, J_z\}$$

$$j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \ldots \oplus |j_1 - j_2|$$

$$|j_1 j_2; jm\rangle = \sum_{m_1+m_2=m} |j_1 j_2; m_1 m_2\rangle \underbrace{\langle j_1 j_2; m_1 m_2 | j_1 j_2; jm\rangle}_{\text{Clebsch}-\text{Gordan coefficient}}$$

$$\vec{J}_1 \cdot \vec{J}_2 = \frac{1}{2}(J_{1+}J_{2-} + J_{1-}J_{2+}) + J_{1z}J_{2z}$$

$$\text{Combining two spin 1/2 :} \quad \frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

$$\begin{aligned} |\,1,1\,\rangle \; &= |\uparrow\uparrow\rangle\,, \\ |\,1,0\,\rangle \; &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\,, \qquad |0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \\ |1,-1\rangle &= |\downarrow\downarrow\rangle\,. \end{aligned}$$